GLOBAL ATTRACTORS OF NONLOCAL REACTION DIFFUSION EQUATIONS WITH EXPONENTIAL NONLINEARITIES

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Abstract: *In this paper, we investigate the existence, uniqueness, and continuity of weak solutions with respect to initial values for a nonlinear parabolic equation of reaction-diffusion nonlocal type by an application of the Faedo-Galerkin approximation and Aubin-Lions- Simon compactness results. The nonlocal quantity appears in the diffusion coefficient. Moreover, we deal with a new class of nonlinearities which is no restriction on the growth of the nonlinearities. The long -time behaviour of solutions to that problem is considered via the concept of global attractors for the associated semigroups.*

Keywords: *Nonlocal reaction diffusion equation, weak solution, nonlocal type, global attractors, exponential nonlinearity.*

1. Introduction

Let $\Omega \subset \mathbb{R}^n, n \geq 1$, be a bounded open set with a sufficiently smooth boundary $\partial\Omega$. We are concerned with the following initial boundary valued problem

$$\frac{\partial u}{\partial t} - a(|u|_2^2)\Delta u + f(u) = g(x), \quad x \in \Omega, t > 0,$$

$$u(x,t) = 0, \quad x \in \partial\Omega, t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in \Omega,$$

where the nonlinearity f, the external force g and the diffusion coefficient a satisfy the following conditions:

(H_1) $a \in C(\mathbb{R}, \mathbb{R}_+)$ isLipschitz continuous in the sense that there exists a constant L such that

$$|a(t) - a(s)| \leq L|t - s|, \quad \forall t, s \in \mathbb{R}, \tag{1.2}$$

and bounded, i.e, there are two positive constants m, M such that

$$0 < m \leq a(t) \leq M, \quad \forall t \in \mathbb{R}, \tag{1.3}$$

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(H_2) $f:\mathbb{R}\to\mathbb{R}$ is a continuously differentiable function satisfying

$$f(u)u \geq -\mu u^2 - c_1,\ f'(u) \geq -\alpha, \tag{1.4}$$

where c_1, α are two positive constants, $0 < \frac{\mu}{m} < \lambda_1$ and λ_1 is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$.

$$(H_3)\ g \in L^2(\Omega). \tag{1.5}$$

During the last decade, the nonlinear parabolic equations with nonlocal terms have been extensively studied associated with many operators for various issues and applications such as in physics, in fluid mechanics, in financial mathematics, in population dynamics, etc. One of the justification of such models is the fact that in reality the measurements are not made pointwise, but through some local average. For more details, we refer to, for instance, [2], [3], [6], [7], [8], [9] and in the references therein. In recent years, many mathmaticians have been studying problems associated with the Laplacian operator which appears in a variety of physical fields (see for example [2], [6], [8]). Usually, there are two main kinds of nonlinearities which have been considered (see [2], [6]). The first one is the class of nonlinearities that is locally Lipschitzian continuous and satisfies a Sobolev growth condition

$$\begin{aligned} &|f(u)| \leq c(1+|u|^{\rho}), \quad \rho < \frac{n}{n-2} \\ &f(u)u \geq -\mu u^2 - c, \\ &f'(u) \geq -\alpha, \end{aligned}$$

The second one is the class of nonlinearities that satisfies a polynomial growth

$$\begin{aligned} &c_1|u|^p - c_0 \leq f(u)u \leq c_2|u|^p + c_0, \\ &f'(u) \geq -\alpha, \end{aligned}$$

for some $p \geq 2$. Note that for both the above classes of nonlinearities require some restriction on the upper growth of the nonlinearities imposed which an exponential nonlinearity, for example, $f(u) = e^u$, does not hold. In this paper, we will relax the condition on f in order to remove this restriction. We will consider the problem (1.1) with the homogeneous Dirichlet boundary condition, in which the diffusion coefficient a depends on the L^2-norm of the solution (see [2], [3], [6], [7] for more types of the nonlocal diffusion coefficient), the nonlinearity satisfies an exponential growth type condition and the external force g belongs to $L^2(\Omega)$.

The problem (1.1) contains some important classes of parabolic equations, such as the semilinear heat equations (when $a = const > 0$), the Laplacian equation (when $a = 1$), etc. The existence and long-time behaviour of solutions to these equations have attracted interest in recent years.

The structure of the paper is organized as follows. In section 2, we prove the existence, uniqueness, continuity and joint continuity of weak solutions with respect to the initial values by using the compactness method and weak convergence techniques in [2]. In section 3, we prove the existence of global attractors for the semigroup generated by the problem in various spaces. The main novelty of the paper is that the nonlinearity can grow exponentially.

Before to start, let us introduce some notation that will be used in the sequel. As usual, the inner product in $L^2(\Omega)$ will be denoted by $(.,.)$ and by $|.|_2$ its associated norm. The inner product in $H_0^1(\Omega)$ is presented by $((.,.))$ and by $\|\ \|_2$ its associated norm. By $\langle .,. \rangle$, we represent the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ andby $\|\ \|_*$ the norm in $H^{-1}(\Omega)$. We identify $L^2(\Omega)$ with its dual, and so, we have a chain of compact and dense embeddings $H_0^1(\Omega) \subset\subset L^2(\Omega) \subset H^{-1}(\Omega)$. We use C to denote various constants whose values may change with each appearance.

2. Existence and uniqueness of weak solutions

In this section, we will study the existence and uniqueness of weak solution to (1.1). It is worth if we first give the definition of weak solution of our problem. In what follows, we assume that the initial data $u_0 \in L^2(\Omega)$ is given.

Definition 2.1. A weak solution to (1.1) is a function u that, for all $T > 0$, belongs to $L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$, $f(u) \in L^1(\Omega_T)$, $u(0) = u_0$ and such that for all $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we have

$$(\frac{d}{dt}u(t),v) + a(|u|_2^2)((u(t),v)) + \langle f(u),v \rangle = (g,v), \tag{2.1}$$

where $\Omega_T = \Omega \times (0,T)$ and the previous equation must be understood in the sense of $\mathcal{D}'(0,T)$.

It is known that (see [1]) that if $u \in V$ and $\frac{\partial u}{\partial t} \in V^*$, then $u \in C([0,T];L^2(\Omega))$. This makes the initial condition in problem (1.1) meaningful. The existence of weak solution is assured by the following theorem

Theorem 2.1. Let $u_0 \in L^2(\Omega)$ and $0 < T < +\infty$. Assume (H_1), (H_2), and (H_3) hold. Then problem (1.1) has a unique weak solution on the interval $(0,T)$, i.e, there exists a function u such that

$$u \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega)),\ u_t \in L^2(0,T;H^{-1}(\Omega)),\ u(0) = u_0,$$

$$\frac{d}{dt}(u,v)+a(|u|_2^2)((u,v))+\langle f(u),v\rangle=(g,v), \tag{2.2}$$

forall $v\in H_0^1(\Omega)\cap L^\infty(\Omega)$, where (2.2) must be understood as an equality in $\mathcal{D}'(0,T)$. Moreover, the mapping $u_0\to u(t)$ is continuous on $L^2(\Omega)$.

Proof

i) Existence. Due to the theory of ordinary differential equations in variant t, we can find, for each integer $n\geq 1$, the Galerkin approximated solution by the following form

$$u_n(t)=\sum_{j=1}^{n}u_{nj}(t)w_j, \tag{2.3}$$

where $\{w_j; j\geq 1\}\subset H_0^1(\Omega)\cap L^\infty(\Omega)$ is a Hilbert basis of $L^2(\Omega)$ such that $\bigcup_{n\in\mathbb{N}}\text{span}\{w_1,w_2,\cdots,w_n\}$ is dense in $H_0^1(\Omega)\cap L^\infty(\Omega)$, and $u_{nj}(t)$ are solutions of the following problem

$$\frac{d}{dt}(u_n(t),w_j)+a(|u_n|_2^2)((u_n(t),w_j))+\langle f(u_n(t)),w_j\rangle=(g,w_j), \tag{2.4}$$

$$(u_n(0),w_j)=(u_0,w_j).$$

Now, multiplying by $u_{nj}(t)$ in (2.4), summing from $j=1$ to n. We obtain

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|_2^2+a(|u_n|_2^2)\|u_n(t)\|^2+\int_\Omega f(u_n(t))u_n(t)dx=\int_\Omega gu_n(t)dx. \tag{2.5}$$

Taking (1.4) into account and using the Cauchy inequality, we get the estimate

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|_2^2+a(|u_n|_2^2)\|u_n(t)\|^2-\mu|u_n(t)|_2^2-c_1|\Omega|\leq\frac{1}{2\varepsilon}|g|_2^2+\frac{\varepsilon}{2}|u_n(t)|_2^2, \tag{2.6}$$

since λ_1 is the first eigen value of $(-\Delta,H_0^1(\Omega))$ satisfying $0<\dfrac{\mu}{m}<\lambda_1$. Therefore, in view of (1.3), we deduce

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|_2^2+\Big(m-\frac{\mu}{\lambda_1}-\frac{\varepsilon}{2\lambda_1}\Big)\|u_n(t)\|^2\leq\frac{1}{2\varepsilon}|g|_2^2+c_1|\Omega|, \tag{2.7}$$

with sufficient small ε that makes $m-\dfrac{\mu}{\lambda_1}-\dfrac{\varepsilon}{2\lambda_1}>0$ satisfied. Now, integrating (2.7) between 0 and $t\in(0,T)$, we get

$$|u_n(t)|_2^2+2\Big(m-\frac{\mu}{\lambda_1}-\frac{\varepsilon}{2\lambda_1}\Big)\int_0^t\|u_n(s)\|^2 ds\leq\frac{1}{2\varepsilon}|g|_2^2T+c_1|\Omega|T+|u_0|_2^2. \tag{2.8}$$

This inequality yields

$\{u_n\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$,

$\{u_n\}$ is bounded in $L^2(0,T;H_0^1(\Omega))$.

Note that $-a(|u_n|_2^2)\Delta u_n$ defines an element of $H^{-1}(\Omega)$, given by the duality $\langle -a(|u_n|_2^2)\Delta u_n, w\rangle = a(|u_n|_2^2)\int_\Omega \nabla u_n \nabla w dx,$ for all $w \in H_0^1(\Omega)$. In addition, from (1.3) and the boundedness of $\{u_n\}$ in $L^2(0,T;H_0^1(\Omega))$, we deduce that $\{-a(|u_n|_2^2)\Delta u_n\}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$. From (1.3) and (2.5), we can obtain that

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|_2^2 + m\lambda_1|u_n(t)|_2^2 + \int_\Omega f(u_n(t))u_n(t)dx \le \frac{1}{2\varepsilon}|g|_2^2 + \frac{\varepsilon}{2}|u_n(t)|_2^2 .$$

We choose $\varepsilon = m\lambda_1$, and then this leads to

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|_2^2 + \int_\Omega f(u_n(t))u_n(t)dx \le \frac{1}{2m\lambda_1}|g|_2^2 . \tag{2.9}$$

Integrating (2.9) from 0 to T, we have

$$\frac{1}{2}|u_n(T)|_2^2 + \int_0^T\int_\Omega f(u_n(t))u_n(t)dxdt \le \frac{1}{2m\lambda_1}|g|_2^2\, T + \frac{1}{2}|u_0|_2^2 .$$

The last inequality implies that

$$\int_{\Omega_T} f(u_n(t))u_n(t)dxdt \le C, \tag{2.10}$$

For some positive constant C, we define $h(u_n) = f(u_n) + \nu u_n$, where $\nu > \mu$. In view of (1.4), it is easily to prove that $h(u_n)u_n + c_1 \ge 0$ for all $u_n \in \mathbb{R}$, we have

$$\begin{aligned}
&\int_{\Omega_T}|h(u_n(t))|dxdt = \int_{\Omega_T\cap\{|u_n|>1\}}|h(u_n(t))|dxdt + \int_{\Omega_T\cap\{|u_n|\le 1\}}|h(u_n(t))|dxdt\\
&\le \int_{\Omega_T\cap\{|u_n|>1\}}|h(u_n(t))u_n(t)|dxdt + \int_{\Omega_T\cap\{|u_n|\le 1\}}|h(u_n(t))|dxdt\\
&\le \int_{\Omega_T\cap\{|u_n|>1\}}|h(u_n(t))u_n(t)+c_1|dxdt + \int_{\Omega_T\cap\{|u_n|>1\}}c_1dxdt + \int_{\Omega_T\cap\{|u_n|\le 1\}}|h(u_n(t))|dxdt\\
&\le \int_{\Omega_T}|h(u_n(t))u_n(t)+c_1|dxdt + c_1|\Omega_T| + \sup_{|s|\le 1}|h(s)||\Omega_T|\\
&= \int_{\Omega_T}h(u_n(t))u_n(t)+c_1dxdt + c_1|\Omega_T| + \sup_{|s|\le 1}|h(s)||\Omega_T|\\
&= \int_{\Omega_T}f(u_n(t))u_n(t)dxdt + \nu\int_{\Omega_T}u_n(t)^2dxdt + 2c_1|\Omega_T| + \sup_{|s|\le 1}|h(s)||\Omega_T| \le C,
\end{aligned}$$

since $\{u_n\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$, Ω is bounded, and combining with (2.10), we deduce that $h(u_n)$ is bounded in $L^1(\Omega_T)$, and so is $f(u_n)$. As a consequence, there exists

$u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, $\xi_1 \in L^1(\Omega_T)$ and $\xi_2 \in L^2(0,T;H^{-1}(\Omega))$, and a subsequence of u_n (relabelled the same) such that

$$u_n \rightharpoonup^* u \text{ weakly-star in } L^\infty(0,T;L^2(\Omega)),$$

$$u_n \rightharpoonup u \text{ in } L^2(0,T;H_0^1(\Omega)),$$

$$f(u_n) \rightharpoonup \xi_1 \text{ in } L^1(\Omega_T), \tag{2.11}$$

$$-a(|u_n|_2^2)\Delta u_n \rightharpoonup \xi_2 \text{ in } L^2(0,T;H^{-1}(\Omega)), \tag{2.12}$$

for all $T > 0$. We will show that $\xi_1 = f(u)$ and $\xi_2 = -a(|u|_2^2)\Delta u$ by using the compactness method. On the other hand, $\frac{du_n}{dt} = a(|u_n|_2^2)\Delta u_n - f(u_n) + g$ plays a role as an operator on $H_0^1(\Omega) \cap L^\infty(\Omega)$. We deduce that $\{\frac{du_n}{dt}\}$ is bounded in $L^2(0,T;H^{-1}(\Omega)) + L^1(\Omega_T)$, and therefore in $L^1(0,T;H^{-1}(\Omega) + L^1(\Omega))$. As far as we know $H_0^1(\Omega) \subset\subset L^2(\Omega) \subset H^{-1}(\Omega) + L^1(\Omega)$. By the Aubin - Lions - Simon compactness lemma (see [5]), we have that $\{u_n\}$ is compact in $L^2(0,T;L^2(\Omega))$. In view of Lemme1.3, p.12 in [4], we identify ξ_1 and ξ_2 in (2.11) and (2.12) respectively,

$$f(u_n) \rightharpoonup f(u) \text{ in } L^1(\Omega_T), \tag{2.13}$$

$$-a(|u_n|_2^2)\Delta u_n \rightharpoonup -a(|u|_2^2)\Delta u \text{ in } L^2(0,T;H^{-1}(\Omega)), \tag{2.14}$$

Then, if we consider fixed n, $\varphi \in \mathcal{D}(0,T)$, and $w \in span\{w_1, w_2, \cdots, w_n\}$, it holds for all $m > n$

$$-\int_0^T (u_m(t), w)\varphi'(t)dt + \int_0^T a(|u_m|_2^2)\langle -\Delta u_m(t), w\rangle \varphi(t)dt + \int_0^T < f(u_m(t)), w > \varphi(t)dt = \int_0^T (g, w)\varphi(t)dt.$$

Now, let m tend to infinity, using (2.13) and (2.14), and compactness of $\{u_n\}$ in $L^2(0,T;L^2(\Omega))$.

$$-\int_0^T (u(t), w)\varphi'(t)dt + \int_0^T a(|u|_2^2)\langle -\Delta u(t), w\rangle \varphi(t)dt + \int_0^T < f(u(t)), w > \varphi(t)dt = \int_0^T (g, w)\varphi(t)dt,$$

for all $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$, since $\bigcup_{n\in\mathbb{N}} \text{span}\{w_1, w_2, \cdots, w_n\}$ is dense in $H_0^1(\Omega) \cap L^\infty(\Omega)$. Therefore, $\frac{du}{dt} - a(|u|_2^2)\Delta u + f(u) = g$, in $\mathcal{D}'(0,T;H^{-1}(\Omega) + L^1(\Omega))$, an taking into account the regularity of u and u', it holds that $u \in C([0,T];L^2(\Omega))$. Finally, we only need to check that $u(0)=u_0$, we also fix $n\ge 1, \varphi\in H^1(0,T)$ such that $\varphi(T)=0$ and $\varphi(0)\neq 0$, and $w\in span\{w_1,w_2,\cdots,w_n\}$, and consider $m>n.$ We have

$$-(u_0,w)\varphi(0)-\int_0^T(u_m(t),w)\varphi'(t)dt+\int_0^T a(|u_m|_2^2)\langle -\Delta u_m(t),w\rangle\varphi(t)dt$$
$$+\int_0^T\langle f(u_m(t)),w\rangle\varphi(t)dt=\int_0^T(g,w)\varphi(t)dt.$$

Let $m\to\infty$

$$\begin{aligned}&-(u_0,w)\varphi(0)-\int_0^T(u(t),w)\varphi'(t)dt+\int_0^T a(|u|_2^2)\langle -\Delta u(t),w\rangle\varphi(t)dt\\&+\int_0^T\langle f(u(t)),w\rangle\varphi(t)dt=\int_0^T(g,w)\varphi(t)dt.\end{aligned}\tag{2.15}$$

On the other hand, from (2.1),

$$\begin{aligned}&-(u(0),w)\varphi(0)-\int_0^T(u(t),w)\varphi'(t)dt+\int_0^T a(|u|_2^2)\langle -\Delta u(t),w\rangle\varphi(t)dt\\&+\int_0^T\langle f(u(t)),w\rangle\varphi(t)dt=\int_0^T(g,w)\varphi(t)dt.\end{aligned}\tag{2.16}$$

Then, comparing (2.15) with (2.16), it holds that $(u_0,w)\varphi(0)=(u(0),w)\varphi(0)$ with $w\in span\{w_1,w_2,\cdots,w_n\}$. This leads to $u(0)=u_0$, and u is a weak solution to problem (1.1).

ii) Uniqueness and continuous dependence on the initial data. Let us denote by u_1 and u_2 two weak solutions of (1.1) with initial data u_{01}, $u_{02}\in L^2(\Omega)$. Then

$$(\frac{d}{dt}u_1,v)+a(|u_1|_2^2)\int_\Omega\nabla u_1\nabla v dx+\langle f(u_1),v\rangle=(g,v),$$

and

$$(\frac{d}{dt}u_2,v)+a(|u_2|_2^2)\int_\Omega\nabla u_2\nabla v dx+\langle f(u_2),v\rangle=(g,v),$$

thus

$$(\frac{d}{dt}(u_1-u_2),v)+a(|u_1|_2^2)\int_\Omega\nabla u_1\nabla v dx-a(|u_2|_2^2)\int_\Omega\nabla u_2\nabla v dx+\langle f(u_1)-f(u_2),v\rangle=0,$$

which leads to

$$(\frac{d}{dt}(u_1-u_2),v)+a(|u_1|_2^2)\int_\Omega\nabla(u_1-u_2)\nabla v dx+\langle\hat f(u_1)-\hat f(u_2),v\rangle$$
$$=(a(|u_2|_2^2)-a(|u_1|_2^2))\int_\Omega\nabla u_2\nabla v dx+\alpha(u_1-u_2,v),$$

where $\hat{f}(s) = f(s) + \alpha s$. Taking $v = (u_1 - u_2)(t)$ for a.e.t, we have

$$\frac{1}{2}\frac{d}{dt}|u_1 - u_2|_2^2 + a(|u_1|_2^2)\int_\Omega |\nabla(u_1 - u_2)|^2 dx + \int_\Omega (\hat{f}(u_1) - \hat{f}(u_2))(u_1 - u_2)dx$$
$$\le |a(|u_2|_2^2) - a(|u_1|_2^2)| \int_\Omega |\nabla u_2||\nabla(u_1 - u_2)|dx + \alpha\int_\Omega |u_1 - u_2|^2 dx.$$

Thanks to (1.4) we have $\int_\Omega (\hat{f}(u_1) - \hat{f}(u_2))(u_1 - u_2) \ge 0.$ So

$$\frac{1}{2}\frac{d}{dt}|u_1 - u_2|_2^2 + a(|u_1|_2^2)\int_\Omega |\nabla(u_1 - u_2)|^2 dx$$
$$\le |a(|u_2|_2^2) - a(|u_1|_2^2)| \int_\Omega |\nabla u_2||\nabla(u_1 - u_2)|dx + \alpha\int_\Omega |u_1 - u_2|^2 dx.$$

Applying the Cauchy - Schwarz inequality and putting this together with (1.2) and (1.3), we get the estimate

$$\frac{1}{2}\frac{d}{dt}|u_1 - u_2|_2^2 + m\| u_1 - u_2\|_2^2 \le L\,\| |u_2|_2^2 - |u_1|_2^2| \| u_2\|_2 \| u_1 - u_2\|_2 + \alpha |u_1 - u_2|_2^2 .$$

Then, applying Young's inequality we obtain

$$\frac{1}{2}\frac{d}{dt}|u_1 - u_2|_2^2 + m\| u_1 - u_2\|_2^2 \le \frac{m}{2}\| u_1 - u_2\|_2^2 + \eta(t)|u_1 - u_2|_2^2,$$

which gives $\frac{d}{dt}|u_1 - u_2|_2^2 \le \eta(t)|u_1 - u_2|_2^2$. Then, with some more computation, we obtain $\sup_{t\in[0,T]} |u_1(t) - u_2(t)|_2 \le C|u_{01} - u_{02}|_2$, where C is some constant which, we will see later, depends on $T, \lambda_1, \mu, m, |\Omega|, c_1, |g|_2^2$. Hence, we get the desired results, i.e, the solution is uniqueness and continuous dependence on the initial data.

3. Global attractors

Thanks to Theorem 2.1, we can define a continuous (nonlinear) semigroup $S(t): L^2(\Omega) \to L^2(\Omega)$ associated to problem (1.1) as follows $S(t)u_0 := u(t, u_0)$, where $u(t, u_0)$ is the unique weak solution of (1.1) with the initial datum u_0. We will prove that the semigroup $S(t)$ has a global attractor $\mathcal{A}$ in $L^2(\Omega)$. For the sake of brevity, in the following important lemmas, we give some formal caculations, the rigorous proof is done by use of Galerkin approximations and Lemma 11.2 in [5].

Lemma 3.1. The semigroup $\{S(t)\}_{t\ge 0}$ has a bounded absorbing set in $L^2(\Omega)$.

Proof. Multiplying (1.1) by u we have

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + a(|u|_u^2)\| u\|_2^2 + \langle f(u), u\rangle = (g, u). \tag{3.1}$$

We perform the similar way as (2.6), (2.7) by using hypotheses (1.2) - (1.5), the Cauchy's inequality and the Gronwall's inequality, we obtain

$$|u(t)|_2^2 \le |u_0|_2^2\, e^{-(m\lambda_1-\mu)t} + R_1,$$

where

$$R_1 = R_1(\lambda_1, \mu, m, |\Omega|, c_1, |g|_2^2) = \frac{2c_1|\Omega|(m\lambda_1-\mu) + |g|_2^2}{(m\lambda_1-\mu)^2}.$$

Therefore, if choosing $\rho_1 = 2R_1$, we are sure that

$$|u(t)|_2^2 \le \rho_1, \tag{3.2}$$

for all $t \ge T_1 = T_1(\lambda_1, \mu, m, |u_0|_2)$, and so the proof is completed.

Lemma 3.2. The semigroup $\{S(t)\}_{t\ge 0}$ has a bounded absorbing set in $H_0^1(\Omega)$.

Proof. Multiplying (1.1) by $-\Delta u$, and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + a(|u|_2^2)|\Delta u|_2^2 = -\int_\Omega f'(u)(\nabla u)^2 dx - \int_\Omega g\Delta u dx \le \alpha\|u\|_2^2 + \frac{1}{2m}|g|_2^2 + \frac{m}{2}|\Delta u|_2^2,$$

Of course, we have already used the Cauchy inequality, and putting this with (1.3), it leads to

$$\frac{d}{dt}\|u\|_2^2 \le 2\alpha\|u\|_2^2 + \frac{1}{m}|g|_2^2. \tag{3.3}$$

On the other hand, integrating (3.1) from t to $t+1$ and using (1.3) and (1.4) and the estimation (3.2)

$$\begin{aligned}\int_t^{t+1}\|u\|_2^2 ds + \frac{1}{2m}|u(t+1)|_2^2 &\le \frac{1}{2m}|u(t)|_2^2 + \frac{c_1}{m}|\Omega|^2 + \frac{\mu+1}{m}\int_t^{t+1}|u|_2^2 ds + \frac{1}{4m}|g|_2^2 \\ &\le \rho_2 = \rho_2(\lambda_1, \mu, m, |\Omega|, C_1, |g|_2^2),\end{aligned} \tag{3.4}$$

for all $t \ge T_1 = T_1(\lambda_1, \mu, m, |u_0|_2)$. By the uniform Gronwall inequality, from (3.3) and (3.4) we deduce that

$$\|u(t)\|_2^2 \le \rho_2, \tag{3.5}$$

for all $t \ge T_2 = T_1 + 1$. The proof is complete.

As a direct consequence of Lemma 3.1, and Lemma 3.2 and the compactness of the embedding $H_0^1(\Omega) \subset\subset L^2(\Omega)$, we get one of the main results of this section.

Theorem 3.1. Suppose that the hypotheses (H_1), (H_2), and (H_3) hold. Then the semigroup $S(t)$ generated by problem (1.1) has a connected global attractor $\mathcal{A}$ in $L^2(\Omega)$.

With more sophisticated arguments, it is possible to show that the regularity of the attractor increases as a becomes more regular.

Lemma 3.3. The semigroup $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $H^2(\Omega)\cap H_0^1(\Omega)$.

Proof. Differentiating the first equation of problem (1.1) with respect to t, then taking the dual product of the resultant with u_t yields

$$\frac{1}{2}\frac{d}{dt}|u_t|_2^2+a(|u|_2^2)|\nabla u_t|_2^2+\int_\Omega f'(u)u_t^2dx=-2a'(|u|_2^2)\int_\Omega uu_tdx\int_\Omega \nabla u\nabla u_tdx.$$

and perform the following estimate deduced from the Holder's inequality

$$\frac{d}{dt}|u_t|_2^2+2m|\nabla u_t|_2^2-2\alpha|u_t|_2^2\leq 4|a'(|u|_2^2)||u|_2|u_t|_2|\nabla u|_2|\nabla u_t|_2. \tag{3.6}$$

We make a use of the estimates (3.2) and (3.5) $(i.e.|u(t)|_2^2\leq\rho_1,\|u(t)\|_2^2\leq\rho_2)$, and we define

$$\gamma=2\sup_{s\leq\rho_1}|a'(s)||u|_2|\nabla u|_2. \tag{3.7}$$

we get from (3.6) and (3.7) that

$$\frac{d}{dt}|u_t|_2^2+2m\|u_t\|_2^2-2\alpha|u_t|_2^2\leq 2\gamma\|u_t\|_2|u_t|_2.$$

Applying the Young's inequality, we get the estimate

$$\frac{d}{dt}|u_t|_2^2+2m\|u_t\|_2^2-2\alpha|u_t|_2^2\leq 2\gamma(\varepsilon\|u_t\|_2^2+\frac{1}{4\varepsilon}|u_t|_2^2). \tag{3.8}$$

The last inequality leads to the following estimation if we choose $0<\varepsilon\leq m$

$$\frac{d}{dt}|u_t|_2^2\leq(2\alpha+\frac{\gamma}{2\varepsilon})|u_t|_2^2. \tag{3.9}$$

Multiplying the first equation in (1.1) by u_t, we obtain

$$|u_t|_2^2+a(|u|_2^2)\int_\Omega \Delta uu_tdx+\int_\Omega f(u)u_tdx=\int_\Omega g(x)u_tdx,$$

so

$$a(|u|_2^2)\int_\Omega \nabla u\nabla u_tdx+\int_\Omega f(u)u_tdx-\int_\Omega g(x)u_tdx=-|u_t|_2^2\leq 0.$$

Using (1.3), we have

$$m\int_\Omega \nabla u\nabla u_tdx+\int_\Omega f(u)u_tdx-\int_\Omega g(x)u_tdx\leq 0.$$

We define $F(s)=\int_0^s f(\sigma)d\sigma.$

Hence

$$\frac{d}{dt}(\frac{m}{2}\|u\|_2^2+\int_\Omega F(u)dx-\int_\Omega g(x)udx)\leq-|u_t|_2^2\leq 0. \tag{3.10}$$

On the other hand, integrating (3.1) from t to $t+1$, and using (3.2), and (1.3), we have

$$\int_t^{t+1}[m\|u\|_2^2+\int_\Omega f(u)udx-\int_\Omega g(x)udx]ds\le\rho_1. \quad (3.11)$$

It follows from (1.5) that

$$F(u)\le f(u)u+\alpha\frac{u^2}{2},\quad \forall u\in\mathbb{R}. \quad (3.12)$$

From (3.11) and (3.12), we get

$$\int_t^{t+1}[m\|u\|_2^2+\int_\Omega f(u)udx-\int_\Omega g(x)udx]ds\ge\int_t^{t+1}[\frac{m}{2}\|u\|_2^2+\int_\Omega F(u)dx-\int_\Omega g(x)udx]ds-\alpha\frac{\rho_1}{2}$$

for all $t\ge T_1$. Thus

$$\int_t^{t+1}[\frac{m}{2}\|u\|_2^2+\int_\Omega F(u)dx-\int_\Omega g(x)udx]ds\le(1+\frac{\alpha}{2})\rho_1. \quad (3.13)$$

Therefore, from (3.10) and (3.13), by using the uniform Gronwallinequality, we obtain

$$\frac{m}{2}\|u\|_2^2+\int_\Omega F(u)dx-\int_\Omega g(x)udx\le\rho_3, \quad (3.14)$$

for all $t\ge T_2=T_1+1$. Integrating (3.10) from t to $t+1$ and using (3.14), we have

$$\int_t^{t+1}|u_t|_2^2\le\rho_3, \quad (3.15)$$

for all $t\ge T_2$. In view of (3.9) and (3.15) and using the uniform Gronwall inequality again, we get

$$|u_t|_2^2\le\rho_3, \quad (3.16)$$

for all $t\ge T_3=T_2+1$. On the other hand, multiplying the first equation in (1.1) by $-\Delta u$, using (1.3) and (1.5), we obtain

$$m|\Delta u|_2^2\le\alpha\|u\|_2^2+|u_t|_2|\Delta u|_2+|g|_2|\Delta u|_2\,. \quad (3.17)$$

Applying the Cauchy inequality, from (3.17), we have

$$m|\Delta u|_2^2\le\alpha\|u\|_2^2+\frac{1}{2\varepsilon}|u_t|_2^2+\frac{1}{2\varepsilon}|g|_2^2+\varepsilon|\Delta u|_2^2\,. \quad (3.18)$$

Taking $\varepsilon=\frac{m}{2}$, it follows from (3.18) that

$$|\Delta u|_2^2\le\frac{2\alpha}{m}\|u\|_2^2+\frac{2}{m^2}|u_t|_2^2+\frac{2}{m^2}|g|_2^2\,. \quad (3.19)$$

Using the estimates (3.5) and (3.15), it implies that

$$|\Delta u|_2^2 \le \rho_4, \tag{3.20}$$

for some constant ρ_4, and all $t \ge T_3$. This completes the proof.

Due to the compactness of the embedding $H^2(\Omega) \cap H_0^1(\Omega) \subset\subset H_0^1(\Omega)$, we get the following important result.

Theorem 3.2. Suppose that the hypotheses (H_1), (H_2), and (H_3) hold. Then the semigroup $S(t)$ generated by problem (1.1) has a global attractor $\mathcal{A}$ in $H^2(\Omega) \cap H_0^1(\Omega)$.

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